Stability inverse problem for diffusion operators

Etibar S. Panakhov[1] and Ahu Ercan[2]

[1]Baku State University, Institute of Applied Mathematics, Baku, Azerbaijan
[2] Firat University, Department of Mathematics, Elazig, Turkey
E-mail: epenahov@hotmail.com[1], ahuduman24@gmail.com[2]

Abstract

In this study, the certain stability problem is studied for diffusion equation by special method on a finite interval. The method which used was given firstly by Ryabushko for Sturm-Liouville problem in [23]. There are many different stability criteria. In practice, any one of a number of different stability criteria is applied. The stability of spectral functions has been shown by using the asymptotics of the eigenvalues for diffusion problems with different initial conditions in this work. Also, we introduce the notion of norming constants and establish their interrelation with the spectra.

2010 Mathematics Subject Classification. **34B09** 34D20. 34L05
Keywords. Stability, spectral function, diffusion equation.

1 Introduction

Inverse problems of spectral analysis are the problems of constructing a linear operator from some of its spectral characteristics. An effective method of constructing Sturm-Liouville operator from two spectra is given in [3]. Levitan [4] established determination of the Sturm-Liouville operator from one and two spectra. Jaulent and Jean [6] stated the actual background of diffusion operators and discussed the inverse problem for the diffusion operator. Gasymov and Guseinov [1] studied the spectral theory of diffusion operators. Panakhov and Koyunbakan [12] considered the half inverse problem for the diffusion operators on the interval. Sat and Panakhov studied generalized degeneracy of the kernels for the diffusion operator in [14]. Direct and inverse problems for the diffusion operator were considered in several works [6]- [14]. In this paper, we shall establish certain stability of spectral functions of two spectral problems for diffusion operators on a finite interval. The stability problem means that when the eigenvalues of the considered problems finitely coincide then the bound of variation for the spectral functions is estimated. In study [23], the author showed the proximity of the spectral functions for regular Sturm-Liouville problems on a finite interval. This type of problem for a self-adjoint operator is considered by Marchenko [18] on a semi-axis. There are many researches about the stability of the spectral theory in the literature [15]- [25]. In this work, the coefficients $p(x)$ and $q(x)$ of quadratic pencils of Sturm-Liouville operators are uniquely determined by two spectra using classic way for Sturm-Liouville operators [2, 5]. The stability problem for diffusion equation is discussed on finite interval using the method given in study [23]. We show the proximity of the spectral functions for diffusion equations.

2 Preliminaries

Consider the diffusion equation

$$-y''(x) + (2\lambda p(x) + q_1(x))\, y(x) = \lambda^2 y(x) \quad , 0 \leq x \leq \pi \tag{2.1}$$

Tbilisi Mathematical Journal Special Issue (IECMSA-2019), pp. 99–109.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: 25 February 2020.
Accepted for publication: 10 April 2020.

with the boundary conditions

$$y'(0) - h_1 y(0) = 0, \quad y'(\pi) + Hy(\pi) = 0 \tag{2.2}$$

$$y'(0) - h_2 y(0) = 0, \quad y'(\pi) + Hy(\pi) = 0 \tag{2.3}$$

where λ is a spectral parameter, $p(x) \in W_2^1[0,\pi]$ and $q_1(x) \in L_2[0,\pi]$, $p(x)$ and $q_1(x)$ are real-valued functions, h_1, h_2 and H are real numbers with $h_1 \neq h_2$. Let us denote the eigenvalues of the boundary value problems (2.1)-(2.2) and (2.1), (2.3) by $\{\lambda_{1,n}\}_{-\infty}^{\infty}$ and $\{\mu_{1,n}\}_{-\infty}^{\infty}$, respectively. The asymptotic expressions for the eigenvalues are given by [1]

$$\lambda_{1,n} = n + c_0 + \frac{c_1}{n} + \frac{c_{1,n}}{n}, \tag{2.4}$$

$$\mu_{1,n} = n + c_0 + \frac{\tilde{c}_1}{n} + \frac{\tilde{c}_{1,n}}{n}, \tag{2.5}$$

for large negative and positive values of n, respectively, where

$$c_0 = \frac{1}{\pi}\int_0^{\pi} p(x)\,dx, \ \sum_n |c_{1,n}|^2 < \infty, \ \sum_n |\tilde{c}_{1,n}|^2 < \infty,$$

$$c_1 = \frac{1}{\pi}\left(h_1 + H + \frac{1}{2}\int_0^{\pi}\left(q_1(x) + p^2(x)\right)dx\right),$$

$$\tilde{c}_1 = \frac{1}{\pi}\left(h_2 + H + \frac{1}{2}\int_0^{\pi}\left(q_1(x) + p^2(x)\right)dx\right).$$

Now consider the second equation

$$-y''(x) + (2\lambda p(x) + q_2(x))\,y(x) = \lambda^2 y(x) \quad , 0 \le x \le \pi \tag{2.6}$$

with the boundary conditions (2.2) and (2.3), where $q_2(x)$ is real-valued function and $q_2(x) \in L_2[0,\pi]$. Denote by$\{\lambda_{2,n}\}_{-\infty}^{\infty}$ and $\{\mu_{2,n}\}_{-\infty}^{\infty}$ the eigenvalues of problems (2.2), (2.6), and (2.3), (2.6), respectively, and the following asymptotic formulas are hold [1]

$$\lambda_{2,n} = n + c_0 + \frac{c_2}{n} + \frac{c_{2,n}}{n}, \tag{2.7}$$

$$\mu_{2,n} = n + c_0 + \frac{\tilde{c}_2}{n} + \frac{\tilde{c}_{2,n}}{n}, \tag{2.8}$$

where

$$\sum_n |c_{2,n}|^2 < \infty, \ \sum_n |\tilde{c}_{2,n}|^2 < \infty,$$

$$c_2 = \frac{1}{\pi}\left(h_1 + H + \frac{1}{2}\int_0^{\pi}\left(q_2(x) + p^2(x)\right)dx\right),$$

$$\tilde{c}_2 = \frac{1}{\pi}\left(h_2 + H + \frac{1}{2}\int_0^{\pi}\left(q_2(x) + p^2(x)\right)dx\right).$$

$\varphi(x,\lambda)$ is the solution of the equation (2.1) satisfying initial conditions

$$\varphi(0,\lambda)=1\ ,\ \ \varphi'(0,\lambda)=h_1. \tag{2.9}$$

Obviously, $\varphi_n(x)=\varphi(x,\lambda_n)$ is an eigenfunction of (2.1)-(2.2), corresponding to the eigenvalue λ_n and the solution of the problem (2.1)-(2.2) is given by [1]

$$\varphi(x,\lambda)=\cos(\lambda x-\alpha_{h_1}(x))+\int_0^x A_{h_1}(x,t)\cos\lambda t dt+\int_0^x B_{h_1}(x,t)\sin\lambda t dt$$

where

$$\alpha_{h_1}(x)=xp(0)+2\int_0^x (A_{h_1}(\xi,\xi)\sin\alpha(\xi)-B_{h_1}(\xi,\xi)\cos\alpha(\xi))\,d\xi.$$

Let's set the norming constants of problem (2.1)-(2.2)

$$\alpha_{1,n}=\int_0^\pi \varphi_n^2(x)\,dx-\frac{1}{\lambda_n}\int_0^\pi p(x)\,\varphi_n^2(x)\,dx.$$

Similarly, $\zeta(x,\lambda)$ is the solution of the equation (2.6) satisfying conditions (2.9).
Then denote the norming constants of problem (2.6), (2.2) as follow:

$$\alpha_{2,n}=\int_0^\pi \zeta_n^2(x)\,dx-\frac{1}{\lambda_n}\int_0^\pi p(x)\,\zeta_n^2(x)\,dx$$

where $\zeta(x,\lambda_n)=\zeta_n(x)$.
Now we set the spectral functions of problems (2.1)-(2.2) and (2.2), (2.6) as follows:

$$\rho_1(\lambda)=\begin{cases}\sum_{\lambda_{1,n}<\lambda}\frac{1}{\alpha_{1,n}}, & \lambda\geq 0\\ -\sum_{\lambda<\lambda_{1,n}}\frac{1}{\alpha_{1,n}}, & \lambda<0\end{cases}$$

$$\rho_2(\lambda)=\begin{cases}\sum_{\lambda_{2,n}<\lambda}\frac{1}{\alpha_{2,n}}, & \lambda\geq 0\\ -\sum_{\lambda<\lambda_{2,n}}\frac{1}{\alpha_{2,n}}, & \lambda<0\end{cases}$$

respectively.

3 Main results

In this section, our purpose is to give effective methods of restoring the diffusion equation from two spectra following the method for classical Sturm-Liouville operator and is to prove the stability theorem for diffusion equation.

Theorem 3.1. We have

$$\alpha_{1,n}=\frac{h_2-h_1}{2\lambda_{1,n}(\mu_{1,n}-\lambda_{1,n})}\prod_{k=-\infty}^{\infty}{}'\left(\frac{\lambda_{1,k}-\lambda_{1,n}}{\mu_{1,k}-\lambda_{1,n}}\right),\ (n=0,\pm1,\pm2,...),$$

where the symbols $\prod'$ denotes the number $k=n$ has been omitted from the infinite product.

Proof. Denote by $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ the solutions of the equation (2.1) which satisfies the boundary conditions

$$\varphi(0,\lambda)=1\ ,\ \ \varphi'(0,\lambda)=h_1,$$
$$\psi(0,\lambda)=1\ ,\ \psi'(0,\lambda)=h_2.$$

The eigenvalues of boundary problems (2.1)-(2.2) and (2.1), (2.3) coincide with the zeros of the functions

$$\Phi_1(\lambda)=\varphi'(\pi,\lambda)+H\varphi(\pi,\lambda),$$
$$\Phi_2(\lambda)=\psi'(\pi,\lambda)+H\psi(\pi,\lambda),$$

respectively. Put

$$f(x,\lambda)=\psi(x,\lambda)+m(\lambda)\varphi(x,\lambda)$$

and require that $f(x,\lambda)$ should hold the condition

$$f'(\pi,\lambda)+Hf(\pi,\lambda)=0.$$

It follows that

$$m(\lambda)=-\frac{\psi'(\pi,\lambda)+H\psi(\pi,\lambda)}{\varphi'(\pi,\lambda)+H\varphi(\pi,\lambda)}=-\frac{\Phi_2(\lambda)}{\Phi_1(\lambda)}.$$

It can be seen from this formula that $m(\lambda)$ is a metamorphic function, its poles and zeros coincide with the eigenvalues of the problems (2.1)-(2.2) and (2.1), (2.3), respectively. On the other hand, it follows from Green's formula that

$$\left(\lambda_1^2-\lambda_2^2\right)\int_0^\pi f(x,\lambda_1)f(x,\lambda_2)\,dx=$$

$$\left(f'(x,\lambda_2)f(x,\lambda_1)-f'(x,\lambda_1)f(x,\lambda_2)\right)\Big|_0^\pi+\int_0^\pi f(x,\lambda_1)f(x,\lambda_2)(2\lambda_1-2\lambda_2)p(x)\,dx$$

$$=(m(\lambda_1)-m(\lambda_2))(h_1-h_2)+2(\lambda_1-\lambda_2)\int_0^\pi f(x,\lambda_1)f(x,\lambda_2)p(x)\,dx. \qquad (3.1)$$

Putting $\lambda_1=\lambda,\ \lambda_2=\overline{\lambda}$ in (3.1) gives

$$\int_0^\pi |f(x,\lambda)|^2\,dx-\frac{1}{Re\lambda}\int_0^\pi p(x)|f(x,\lambda)|^2\,dx=\frac{(h_1-h_2)\,Imm(\lambda)}{2Im\lambda Re\lambda}.$$

It can be seen from this formula that if $h_1>h_2$ then the function $m(\lambda)$ maps the upper half-line onto itself (for $h_1<h_2$ this holds for the lower half-plane). Hence, the zeros and poles of the function $m(\lambda)$ alternates.

Applying Green's formula again, gives

$$\left(\lambda^2-\lambda_n^2\right)\int_0^\pi f(x,\lambda)\varphi(x,\lambda_n)\,dx=h_2-h_1+2(\lambda-\lambda_n)\int_0^\pi f(x,\lambda)\varphi(x,\lambda_n)p(x)\,dx,$$

or

$$\left(\lambda^2-\lambda_n^2\right)\int_0^\pi f(x,\lambda)\varphi(x,\lambda_n)\,dx-2(\lambda-\lambda_n)\int_0^\pi f(x,\lambda)\varphi(x,\lambda_n)p(x)\,dx=h_2-h_1.$$

On the other hand

$$\begin{aligned}&\left(\lambda^2-\lambda_n^2\right)\int_0^\pi \psi\left(x,\lambda\right)\varphi\left(x,\lambda_n\right)dx+\left(\lambda^2-\lambda_n^2\right)m\left(\lambda\right)\int_0^\pi \varphi\left(x,\lambda\right)\varphi\left(x,\lambda_n\right)dx\\&-2\left(\lambda-\lambda_n\right)\int_0^\pi \psi\left(x,\lambda\right)\varphi\left(x,\lambda_n\right)p\left(x\right)dx-2\left(\lambda-\lambda_n\right)m\left(\lambda\right)\int_0^\pi \varphi\left(x,\lambda\right)\varphi\left(x,\lambda_n\right)p\left(x\right)dx\\&=h_2-h_1.\end{aligned}$$

Assuming here that instead of all λ putting λ_n, $(\lambda\to\lambda_n)$

$$\begin{aligned}\alpha_n&=\int_0^\pi \varphi^2\left(x,\lambda_n\right)dx-\frac{1}{\lambda_n}\int_0^\pi p\left(x\right)\varphi^2\left(x,\lambda_n\right)dx\\&=\frac{h_2-h_1}{\lim\limits_{\lambda\to\lambda_n}2\lambda_n\left(\lambda-\lambda_n\right)m\left(\lambda\right)}.\end{aligned}\tag{3.2}$$

It is generally known that the functions of order 1/2 and are determined by their zeros up to constant factors thus:

$$\Phi_1\left(\lambda\right)=C_1\prod_{k=-\infty}^{\infty}\left(1-\frac{\lambda}{\lambda_k}\right),\ \Phi_2\left(\lambda\right)=C_2\prod_{k=-\infty}^{\infty}\left(1-\frac{\lambda}{\mu_k}\right)$$

where C_1 and C_2 are constants. It follows that from these formulas and (3.2)

$$\alpha_n=\frac{h_2-h_1}{2\lambda_n\left(\mu_n-\lambda_n\right)}\frac{C_1}{C_2}\prod_{k=-\infty}^{\infty}\left(\frac{\mu_k}{\lambda_k}\right)\prod_{k=-\infty}^{\infty}{}'\left(\frac{\lambda_k-\lambda_n}{\mu_k-\lambda_n}\right).$$

Now it has to be shown that

$$\frac{C_1}{C_2}\prod_{k=-\infty}^{\infty}\left(\frac{\mu_k}{\lambda_k}\right)=1.\tag{3.3}$$

Note that it follows from the asymptotics formulas for a solution of the diffusion equation that

$$\begin{aligned}&\Phi_1\left(\lambda\right)=\varphi'\left(\pi,\lambda\right)+H\varphi\left(\pi,\lambda\right)\\&=\left(\alpha'_{h_1}\left(\pi\right)-\lambda\right)\sin\left(\lambda\pi-\alpha_{h_1}\left(\pi\right)\right)+A_{h_1}\left(\pi,\pi\right)\cos\lambda\pi\\&+B_{h_1}\left(\pi,\pi\right)\sin\lambda\pi+H\cos\left(\lambda\pi-\alpha_{h_1}\left(\pi\right)\right)\\&+\int_0^\pi\left(HA_{h_1}\left(\pi,t\right)+\left.\frac{\partial A_{h_1}(x,t)}{\partial x}\right|_{x=\pi}\right)\cos\lambda t dt\\&+\int_0^\pi\left(HB_{h_1}\left(\pi,t\right)+\left.\frac{\partial B_{h_1}(x,t)}{\partial x}\right|_{x=\pi}\right)\sin\lambda t dt\end{aligned}$$

and

$$\begin{aligned}&\Phi_2\left(\lambda\right)=\psi'\left(\pi,\lambda\right)+H\psi\left(\pi,\lambda\right)\\&=\left(\alpha'_{h_2}\left(\pi\right)-\lambda\right)\sin\left(\lambda\pi-\alpha_{h_2}\left(\pi\right)\right)+A_{h_2}\left(\pi,\pi\right)\cos\lambda\pi\\&+B_{h_2}\left(\pi,\pi\right)\sin\lambda\pi+H\cos\left(\lambda\pi-\alpha_{h_2}\left(\pi\right)\right)\\&+\int_0^\pi\left(HA_{h_2}\left(\pi,t\right)+\left.\frac{\partial A_{h_2}(x,t)}{\partial x}\right|_{x=\pi}\right)\cos\lambda t dt\\&+\int_0^\pi\left(HB_{h_2}\left(\pi,t\right)+\left.\frac{\partial B_{h_2}(x,t)}{\partial x}\right|_{x=\pi}\right)\sin\lambda t dt.\end{aligned}$$

The last two equalities yield that

$$\lim_{\lambda\to-\infty}\frac{\Phi_1\left(\lambda\right)}{\Phi_2\left(\lambda\right)}=1$$

that is

$$\lim_{\lambda\to-\infty}\frac{C_1}{C_2}\prod_{k=-\infty}^{\infty}\left(1-\frac{\lambda}{\lambda_k}\right)\left(1-\frac{\lambda}{\mu_k}\right)^{-1}$$

$$=\frac{C_1}{C_2}\prod_{k=-\infty}^{\infty}\frac{\mu_k}{\lambda_k}\lim_{\lambda\to-\infty}\prod_{k=-\infty}^{\infty}\frac{\lambda_k-\lambda}{\mu_k-\lambda}=1.$$

We now prove that

$$\lim_{\lambda\to-\infty}\prod_{k=-\infty}^{\infty}\frac{\lambda_k-\lambda}{\mu_k-\lambda}=\lim_{\lambda\to-\infty}\prod_{k=-\infty}^{\infty}\left(1+\frac{\lambda_k-\mu_k}{\mu_k-\lambda}\right)=1. \tag{3.4}$$

From the asymptotic formulas for the eigenvalues of the considered problem, it is known that $\lambda_k=n+c_0+O(1)$ and $\mu_k=n+c_0+O(1)$. Hence $\lambda_k-\mu_k=O(1)$ and the series

$$\sum_{k=-\infty}^{\infty}\frac{\lambda_k-\mu_k}{\mu_k-\lambda}=\sum_{k=1}^{\infty}\left(\frac{\lambda_k-\mu_k}{\mu_k-\lambda}+\frac{\lambda_{-k}-\mu_{-k}}{\mu_{-k}-\lambda}\right)+\frac{\lambda_0-\mu_0}{\mu_0-\lambda}.$$

converges uniformly in a neighbourhood of the point $\lambda=-\infty$. Therefore the limit in each term of the infinite product (3.4) can be approached, that is,

$$\lim_{\lambda\to-\infty}\prod_{k=-\infty}^{\infty}\left(1+\frac{\lambda_k-\mu_k}{\mu_k-\lambda}\right)=1$$

and the Eq. (3.3) is obtained from the last equality. Thereby the proof is completed. Q.E.D.

Then the expressions of norming constants in terms of two spectra of problems (2.1)-(2.2) and (2.1), (2.3), respectively, are shown by

$$\alpha_{1,n}=\frac{h_2-h_1}{2\lambda_{1,n}\left(\mu_{1,n}-\lambda_{1,n}\right)}\prod_{k=-\infty}^{\infty}{}'\left(\frac{\lambda_{1,k}-\lambda_{1,n}}{\mu_{1,k}-\lambda_{1,n}}\right),\quad (n=0,\pm1,\pm2,...),$$

$$\alpha_{2,n}=\frac{h_2-h_1}{2\lambda_{2,n}\left(\mu_{2,n}-\lambda_{2,n}\right)}\prod_{k=-\infty}^{\infty}{}'\left(\frac{\lambda_{2,k}-\lambda_{2,n}}{\mu_{2,k}-\lambda_{2,n}}\right),\quad (n=0,\pm1,\pm2,...).$$

Now we give main theorem in this study.

Theorem 3.2. Let the eigenvalues $\{\lambda_{j,k}\}$ and $\{\mu_{j,k}\}$ $(j=1,2)$ coincide the numbers of $2N+2$ such that $\lambda_{1,k}=\lambda_{2,k}$ and $\mu_{1,k}=\mu_{2,k}$ for $k=1,2,...,N+1$ and $k=-1,...,-N-1$ then

$$\underset{\substack{-\infty<\lambda<\frac{N}{2}\\ -\frac{N}{2}<\lambda<\infty}}{Var}\left\{\rho_1(\lambda)-\rho_2(\lambda)\right\}\le\begin{cases}Ce^{C}\rho_1\left(\frac{N}{2}\right) & ,\ \lambda>0\\ \tilde{C}e^{\tilde{C}}\rho_2\left(-\frac{N}{2}\right) & ,\ \lambda<0\end{cases}$$

respectively, for $k > N+1$, $n < \frac{N}{2}$, $N > 2A$ and $N > 2\tilde{A}$ in case of $\lambda > 0$ and for $k < -N-1$, $n > -\frac{N}{2}$, $N > 2A$ and $N > 2\tilde{A}$ in case of $\lambda < 0$, where

$$A = \left| \frac{1}{2\pi k} \int_0^\pi (q_2(t) - q_1(t))\, dt + \frac{c_{2,k} - c_{1,k}}{k} \right|,$$

$$\tilde{A} = \left| \frac{1}{2\pi k} \int_0^\pi (q_1(t) - q_2(t))\, dt + \frac{\tilde{c}_{1,k} - \tilde{c}_{2,k}}{k} \right|,$$

$C = \frac{\left(1+\frac{3}{2N}\right)\left(A+\tilde{A}\right)}{N\left(\frac{1}{2}+\frac{7}{4N}+\frac{3}{2N^2}\right)}$ and $\tilde{C} = \frac{\left(1+\frac{5}{2N}\right)\left(A+\tilde{A}\right)}{N\left(\frac{1}{2}+\frac{17}{4N}+\frac{15}{2N^2}\right)}$.

Proof. From the definition of the variation, it follows that

$$\underset{\substack{-\infty < \lambda < \lambda_0 \\ \lambda_{\tilde{0}} < \lambda < \infty}}{Var} \{\rho_1(\lambda) - \rho_2(\lambda)\} \le \begin{cases} \max\limits_{\lambda_n < \lambda_0} \left| 1 - \frac{\alpha_{1,n}}{\alpha_{2,n}} \right| \sum_{\lambda_n < \lambda_0} \frac{1}{\alpha_{1,n}}, & \lambda \ge 0 \\ \max\limits_{\lambda_{\tilde{0}} < \lambda_n} \left| 1 - \frac{\alpha_{2,n}}{\alpha_{1,n}} \right| \sum_{\lambda_{\tilde{0}} < \lambda_n} \frac{1}{\alpha_{2,n}}, & \lambda < 0 \end{cases}$$

$$\le \begin{cases} \max\limits_{\lambda_n < \lambda_0} \left| 1 - \frac{\alpha_{1,n}}{\alpha_{2,n}} \right| \rho_1(\lambda_{N+2}), & \lambda \ge 0 \\ \max\limits_{\lambda_{\tilde{0}} < \lambda_n} \left| 1 - \frac{\alpha_{2,n}}{\alpha_{1,n}} \right| \rho_2(\lambda_{-N-2}), & \lambda < 0 \end{cases} \tag{3.5}$$

for $\lambda_0 < \lambda_{N+2}$ and $\lambda_{-N-2} < \lambda_{\tilde{0}}$, respectively. The assessment of the variation of the spectral functions is reduced to evaluation of the absolute value on the right part of (3.5).
So firstly, we must examine the inequality

$$\max_{n < \frac{N}{2}} \left| 1 - \frac{\alpha_{1,n}}{\alpha_{2,n}} \right| \le \max_{n < \frac{N}{2}} \left| 1 - \prod_{k=N+2}^{\infty} \frac{(\lambda_{1,k} - \lambda_{1,n})(\mu_{2,k} - \lambda_{2,n})}{(\lambda_{2,k} - \lambda_{2,n})(\mu_{1,k} - \lambda_{1,n})} \right|. \tag{3.6}$$

In this case, the eigenvalues coincide only for $k = 1, 2, ..., N+1$, that is $\lambda_{1,k} = \lambda_{2,k}$ and $\mu_{1,k} = \mu_{2,k}$ for $k = 1, 2, ..., N+1$. Considering the infinite product

$$\Psi_1(\lambda_n) = \prod_{k=N+2}^{\infty} \frac{(\lambda_{1,k} - \lambda_{1,n})(\mu_{2,k} - \lambda_{2,n})}{(\lambda_{2,k} - \lambda_{2,n})(\mu_{1,k} - \lambda_{1,n})} \tag{3.7}$$

and taking the logarithm of each side of (3.7), it can be easily obtained that

$$|\ln \Psi_1(\lambda_n)| = \left| \sum_{k=N+2}^{\infty} \ln\left(\frac{\lambda_{1,k} - \lambda_{1,n}}{\lambda_{2,k} - \lambda_{2,n}} \right) + \sum_{k=N+2}^{\infty} \ln\left(\frac{\mu_{2,k} - \lambda_{2,n}}{\mu_{1,k} - \lambda_{1,n}} \right) \right|,$$

$$\le \sum_{k=N+2}^{\infty} \left| \ln\left(1 - \frac{\lambda_{2,k} - \lambda_{1,k}}{\lambda_{2,k} - \lambda_{1,n}}\right) \right| + \sum_{k=N+2}^{\infty} \left| \ln\left(1 - \frac{\mu_{1,k} - \mu_{2,k}}{\mu_{1,k} - \lambda_{2,n}}\right) \right|. \tag{3.8}$$

For $k > N+1$, $n < \frac{N}{2}$, it follows that $\left| \frac{\lambda_{2,k} - \lambda_{1,k}}{\lambda_{2,k} - \lambda_{1,n}} \right| < 1$ and $\left| \frac{\mu_{1,k} - \mu_{2,k}}{\mu_{1,k} - \lambda_{2,n}} \right| < 1$.

From the last inequalities Eq. (3.8) can also be written as

$$|\ln \Psi_1(\lambda_n)| < \sum_{k=N+2}^{\infty} \frac{\left|\frac{\lambda_{2,k}-\lambda_{1,k}}{\lambda_{2,k}-\lambda_{1,n}}\right|}{1-\left|\frac{\lambda_{2,k}-\lambda_{1,k}}{\lambda_{2,k}-\lambda_{1,n}}\right|} + \sum_{k=N+2}^{\infty} \frac{\left|\frac{\mu_{1,k}-\mu_{2,k}}{\mu_{1,k}-\lambda_{2,n}}\right|}{1-\left|\frac{\mu_{1,k}-\mu_{2,k}}{\mu_{1,k}-\lambda_{2,n}}\right|}. \tag{3.9}$$

Considering asymptotic formulas of the eigenvalues (2.4), (2.5), (2.7), (2.8) and after some straightforward computations we can obtain that

$$\left|\frac{\lambda_{2,k}-\lambda_{1,k}}{\lambda_{2,k}-\lambda_{1,n}}\right| = \left|\frac{\frac{1}{2\pi k}\int_0^\pi (q_2(t)-q_1(t))\,dt + \frac{c_{2,k}-c_{1,k}}{k}}{\lambda_{2,k}\left(1-\frac{\lambda_{1,n}}{\lambda_{2,k}}\right)}\right|$$

$$< \frac{A}{\left(k-\frac{1}{2}\right)\left(1-\frac{\lambda_{1,[\![\frac{N}{2}]\!]}}{\lambda_{2,N+2}}\right)} < \frac{A\left(1+\frac{3}{2N}\right)}{N\left(\frac{1}{2}+\frac{7}{4N}+\frac{3}{2N^2}\right)} \tag{3.10}$$

and

$$\left|\frac{\mu_{1,k}-\mu_{2,k}}{\mu_{1,k}-\lambda_{2,n}}\right| < \frac{\left|\frac{1}{2\pi k}\int_0^\pi (q_1(t)-q_2(t))\,dt + \frac{\tilde{c}_{1,k}-\tilde{c}_{2,k}}{k}\right|}{\left|\left(k-\frac{1}{2}\right)\left(1-\frac{\lambda_{2,[\![\frac{N}{2}]\!]}}{\mu_{1,N+2}}\right)\right|}$$

$$< \frac{\tilde{A}\left(1+\frac{3}{2N}\right)}{N\left(\frac{1}{2}+\frac{7}{4N}+\frac{3}{2N^2}\right)}. \tag{3.11}$$

Substituting (3.10) and (3.11) into (3.9), it follows that for $N > 2A$ and $N > 2\tilde{A}$

$$|\ln \Psi_1(\lambda_n)| < \frac{\left(1+\frac{3}{2N}\right)\left(A+\tilde{A}\right)}{N\left(\frac{1}{2}+\frac{7}{4N}+\frac{3}{2N^2}\right)}. \tag{3.12}$$

Considering (3.6) with (3.12) and using serial expansion of exponential function then the proof of Theorem 3.2 is completed for positive eigenvalues.
On the other hand, we consider that

$$\max_{n>-\frac{N}{2}}\left|1-\frac{\alpha_{2,n}}{\alpha_{1,n}}\right| = \max_{n>-\frac{N}{2}}\left|1-\prod_{k=-\infty}^{-N-2}\frac{(\lambda_{2,k}-\lambda_{2,n})(\mu_{1,k}-\lambda_{1,n})}{(\lambda_{1,k}-\lambda_{1,n})(\mu_{2,k}-\lambda_{2,n})}\right|. \tag{3.13}$$

In this case, the eigenvalues coincide only for $k = -1, ..., -N-1$, that is $\lambda_{1,k} = \lambda_{2,k}$ and $\mu_{1,k} = \mu_{2,k}$ for $k = -1, ..., -N-1$. Considering the infinite product

$$\Psi_2(\lambda_n) = \prod_{k=-\infty}^{-N-2}\frac{(\lambda_{2,k}-\lambda_{2,n})(\mu_{1,k}-\lambda_{1,n})}{(\lambda_{1,k}-\lambda_{1,n})(\mu_{2,k}-\lambda_{2,n})},$$

by making similar calculations as above it follows that

$$|\ln \Psi_2(\lambda_n)| = \left|\sum_{k=-\infty}^{-N-2}\ln\left(\frac{\lambda_{2,k}-\lambda_{2,n}}{\lambda_{1,k}-\lambda_{1,n}}\right) + \sum_{k=-\infty}^{-N-2}\ln\left(\frac{\mu_{1,k}-\lambda_{1,n}}{\mu_{2,k}-\lambda_{2,n}}\right)\right|$$

$$\leq \sum_{k=-\infty}^{-N-2} \left| \ln \left(1 - \frac{\lambda_{1,k} - \lambda_{2,k}}{\lambda_{1,k} - \lambda_{2,n}} \right) \right| + \sum_{k=-\infty}^{-N-2} \left| \ln \left(1 - \frac{\mu_{2,k} - \mu_{1,k}}{\mu_{2,k} - \lambda_{1,n}} \right) \right|. \tag{3.14}$$

for $k < -N-1$ and $n > -\frac{N}{2}$, it can be easily seen $\left| \frac{\lambda_{2,k} - \lambda_{1,k}}{\lambda_{2,k} - \lambda_{1,n}} \right| < 1$ and $\left| \frac{\mu_{1,k} - \mu_{2,k}}{\mu_{1,k} - \lambda_{2,n}} \right| < 1$. From the last inequalities rewrite Eq. (3.14) as

$$|\ln \Psi_2 (\lambda_n)| < \sum_{k=-\infty}^{-N-2} \frac{\left| \frac{\lambda_{1,k} - \lambda_{2,k}}{\lambda_{1,k} - \lambda_{2,n}} \right|}{1 - \left| \frac{\lambda_{1,k} - \lambda_{2,k}}{\lambda_{1,k} - \lambda_{2,n}} \right|} + \sum_{k=-\infty}^{-N-2} \frac{\left| \frac{\mu_{2,k} - \mu_{1,k}}{\mu_{2,k} - \lambda_{1,n}} \right|}{1 - \left| \frac{\mu_{2,k} - \mu_{1,k}}{\mu_{2,k} - \lambda_{1,n}} \right|}. \tag{3.15}$$

Here using the classical asymptotics of the eigenvalues, it follows that

$$\begin{aligned} \left| \frac{\lambda_{1,k} - \lambda_{2,k}}{\lambda_{1,k} - \lambda_{2,n}} \right| &< \left| \frac{\frac{1}{2\pi k} \int_0^\pi (q_2(t) - q_1(t))\, dt + \frac{c_{2,k} - c_{1,k}}{k}}{\left(-k - \frac{1}{2}\right) \left(1 - \frac{\lambda_{2, \left[\!\left[-\frac{N}{2} \right]\!\right]}}{\lambda_{1,-N-2}} \right)} \right| \\ &< \frac{A \left(1 + \frac{5}{2N}\right)}{N \left(\frac{1}{2} + \frac{17}{4N} + \frac{15}{2N^2}\right)} \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} \left| \frac{\mu_{2,k} - \mu_{1,k}}{\mu_{2,k} - \lambda_{1,n}} \right| &< \frac{\left| \frac{1}{2\pi k} \int_0^\pi (q_1(t) - q_2(t))\, dt + \frac{\tilde{c}_{1,k} - \tilde{c}_{2,k}}{k} \right|}{\left| \left(-k - \frac{1}{2}\right) \left(1 - \frac{\lambda_{1, \left[\!\left[-\frac{N}{2} \right]\!\right]}}{\mu_{2,-N-2}} \right) \right|} \\ &< \frac{\tilde{A} \left(1 + \frac{5}{2N}\right)}{N \left(\frac{1}{2} + \frac{17}{4N} + \frac{15}{2N^2}\right)}. \end{aligned} \tag{3.17}$$

Substituting (3.16) and (3.17) into (3.15), we have for $N > 2A$ and $N > 2\,\tilde{A}$

$$\begin{aligned} |\ln \Psi_2 (\lambda_n)| &< \frac{\frac{A\left(1 + \frac{5}{2N}\right)}{N\left(\frac{1}{2} + \frac{17}{4N} + \frac{15}{2N^2}\right)}}{1 - \frac{A\left(1 + \frac{5}{2N}\right)}{N\left(\frac{1}{2} + \frac{17}{4N} + \frac{15}{2N^2}\right)}} + \frac{\frac{\tilde{A}\left(1 + \frac{5}{2N}\right)}{N\left(\frac{1}{2} + \frac{17}{4N} + \frac{15}{2N^2}\right)}}{1 - \frac{\tilde{A}\left(1 + \frac{5}{2N}\right)}{N\left(\frac{1}{2} + \frac{17}{4N} + \frac{15}{2N^2}\right)}} \\ &< \frac{\left(A + \tilde{A}\right)\left(1 + \frac{5}{2N}\right)}{N \left(\frac{1}{2} + \frac{17}{4N} + \frac{15}{2N^2}\right)}. \end{aligned} \tag{3.18}$$

Then considering (3.13) with (3.18) and using serial expansion of exponential function we can obtain proof of Theorem 3.2 for negative eigenvalues. Therefore the proof is completed. Q.E.D.

4 Conclusion

In spectral theory, the stability theorem explains the stability of solutions or spectral functions of differential equations under small perturbations of initial conditions. For instance, if small perturbations of initial data of considered differential equation lead to small variations then differential equation is stable. In this study, applying Ryabushko's method, we obtain the proximity of spectral functions of problems (2.1)-(2.2) and (2.2), (2.6) whose eigenvalues $\{\lambda_{j,k}\}$ and $\{\mu_{j,k}\}(j = 1, 2)$ coincide numbers of $2N + 2$. Also, the procedure of reconstruction of the equation (2.1) from its two spectra is indicated.

References

[1] M. G. Gasymov and G. Sh. Guseinov, *Determination of a diffusion operator from spectral data*, Akademiya Nauk Azerbaĭdzhansko ĭ SSR. Doklady, **37(2)** (1981) 19–23.

[2] B. M. Levitan and I. S. Sargsjan, *Introduction to spectral theory*, American Mathematical Society, Providence, RI, USA, (1975).

[3] B. M. Levitan and M. G. Gasymov, *Determination of a differential equation by two spectra*, Akademiya Nauk SSSR i Moskovskoe Matematicheskoe Obshchestvo, **19** (1964) 3-63.

[4] B. M. Levitan, *On the determination of the Sturm–Liouville operator from one and two spectra*, Mathematics of the USSR Izvestija., **12** (1978) 179–193.

[5] V. A. Yurko, *An inverse problem for pencils of differential operators*, Matematicheskiĭ Sbornik, **191(10)** (2000), 137–160.

[6] M. Jaulent and C. Jean *The inverse s-wave scattering problem for a class of potentials depending on energy*, Communications in Mathematical Physics, **28** (1972) 177–220.

[7] G. Sh. Guseinov, *On spectral analysis of a quadratic pencil of Sturm-Liouville operators*, Soviet Mathematics Doklady, **32(3)** (1985) 859–862.

[8] I. M. Guseinov and I. M. Nabiev, *The inverse spectral problem for pencils of differential operators*, Sbornik Mathematics, **198(11)** (2007) 1579–1598.

[9] G. Sh. Guseinov, *On construction of a quadratic Sturm-Liouville operator pencil from spectral data,* Proceeding of IMM of NAS of Azerbaijan, **40** (2014) 203–214.

[10] R. Hryniv and N. Pronska, *Inverse spectral problems for energy-dependent Sturm-Liouville equations*, Inverse Problems, **28(8)** (2012) 085008.

[11] R. Kh. Amirov and A. A. Nabiev, *Inverse problems for the quadratic pencil of the Sturm-Liouville equations with impulse*, Abstract and Applied Analysis, Article ID 361989, (2013).

[12] H. Koyunbakan and E. S. Panakhov, *Half-inverse problem for diffusion operators on the finite interval*, Journal of Mathematical Analysis and Applications, **326(2)** (2007) 685–692.

[13] T. Gulsen and E.S. Panakhov, *On the isospectrality of the scalar energy-dependent Schrödinger problems*, Turkish Journal of Mathematics, **42** (2018) 139–154.

[14] M. Sat and E. S. Panakhov, *Spectral problem for diffusion operator*, Applicable Analysis, **93(6)** (2014) 1178–1186.

[15] E. S. Panakhov and R. Yilmazer, *On inverse problem for singular Sturm-Liouville operator from two spectra*, Ukrainian Mathematical Journal, **58(1)** (2006) 147–154.

[16] E. Bas and E. S. Panakhov, *A new approximation for singular inverse Sturm-Liouville Problem*, Thai Journal of Mathematics, **10(3)** (2012) 1024–1030.

[17] V. A. Marchenko, *Spectral theory of Sturm-Liouville operators*, Naukova Dumka, Kiev, (1972) 187–207.

[18] V. A. Marchenko and K. V. Maslov, *Stability of the problem of recovering the Sturm-Liouville operator from the spectral function*, Mathematics of the USSR Sbornik, **81(123)** (1970) 475–502.

[19] J. R. McLaughlin, *Stability theorems for two inverse spectral problems*, Inverse Problems, **4** (1988) 529–540.

[20] A. M. Savchuk and A. A. Shkalikov, *Inverse problems for Sturm–Liouville operators with potentials in Sobolev spaces: Uniform stability*, Functional Analysis and Its Applications, **44** (2010) 270–285.

[21] A. Ercan and E. S. Panakhov, *Stability of the reconstruction discontinuous Sturm-Liouville problem,* Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, **68(1)** (2019) 484–499.

[22] E. S. Panakhov and A. Ercan, *Stability problem of singular Sturm-Liouville equation,* TWMS Journal of Pure and Applied Mathematics, **8(2)** (2017) 148–159.

[23] T. I. Ryabushko, *Stability of the reconstruction of a Sturm–Liouville operator from two spectra, II*, Teor. Funkts. Anal. Prilozhen, **18** (1973) 176–185 (in Russian).

[24] T. I. Ryabushko, *Stability of the reconstruction of a Sturm–Liouville boundary value problem on a semiaxis from two spectra*, Teor. Funktsiĭ Funktsional. Anal. Prilozhen **35** (1981) 96–100.

[25] A. Ercan, *The stability problems in spectral theory of differential operators,* Ph.D. Thesis, Firat University, 2018.